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2-species exclusion processes and combinatorial algebras

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Abstract. Starting from the two-species asymmetric simple exclusion process, we study a subclass of signed permutations, the partially signed permutations, using the combinatorics of Laguerre histories. From this physical and bijective point of view, we obtain natural recoil and descent statistics on partially signed permutations. We define a new combinatorial algebra on these partially signed permutations and show that the segmented composition algebra defined by Novelli and Thibon is a subalgebra of this new algebra. This new point of view allows us to define a new basis for the segmented composition algebra with combinatorial properties. This generalizes classical combinatorial results on permutation and composition algebras.

Keywords: Exclusion processes, bijections, matrix Ansatz, non commutative symmetric functions, signed permutations, algebras

1 Introduction

The two-species asymmetric simple exclusion process (2-ASEP) is a Markov chain with two types of particles (1 and 2) and holes (0). Particles can hop to the right and to the left and particles of type 2 can enter and exit the system. If there are no particles of type 1, we recover the classical ASEP. See Section 3 and [13] for a detailed definition. The stationary distribution of the 2-ASEP gives rise to some interesting combinatorics [2, 8, 7]. In this paper, we interpret it as some generalization of Laguerre histories defined by Françon and Viennot [4]: the *marked Laguerre histories*. We give a bijection between the marked Laguerre histories and a subclass of signed permutations where we do not put a sign on 1. We call them partially signed permutations. For example, the partially signed permutations of size 2 are 12, 21, $1\overline{2}$ and $\overline{2}1$. We denote by B_n the set of signed permutations of size n and by B'_n the set of partially signed permutations. The natural statistic coming from the 2-ASEP define a statistic on a partially signed permutation σ:

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the *Genocchi composition* $GC(\boldsymbol{\sigma})$ and the notion of *patterns* of a partially signed permutation, here the number of occurrences of the pattern $31-2$ and of the pattern $(31,\bar{2})$. Indeed, the GC statistic encodes the state of the Markov chain and the patterns are related to its transition rates. This is related to some combinatorial work of Mandelshtam and Viennot who defined rhombic alternative tableaux [7]. Let us define a segmented composition $\mathbf{I}$ of n as a finite sequence of positive integers of sum n separated by vertical bars or commas. For example, $(1|2|1,2,2)$ is a segmented composition of 8.

Theorem 1. *Given* $\mathbf{I}$ *a segmented composition of* n*, the quotient*

$$\frac{\sum_{\sigma\in B'_n,\mathrm{GC}(\sigma)=\mathbf{I}} q^{\#_{31-2}(\sigma)+\#_{(31,\bar{2})}(\sigma)}}{\sum_{\sigma\in B'_n} q^{\#_{31-2}(\sigma)+\#_{(31,\bar{2})}(\sigma)}} \tag{1.1}$$

is the stationary distribution of the state of the 2-ASEP coded by the segmented composition $\mathbf{I}$*.*

Here $\#_p(\sigma)$ denotes the number of occurrences of the pattern p in σ. Our work here also gives a "permutation" interpretation of the q-statistic that was missing from [8].

Next we generalize the algebra **FQSym** [3] and propose an algebra on B' that we call **PSQSym**. The fundamental basis $F_{\boldsymbol{\sigma}}$ for $\boldsymbol{\sigma} \in B'$ has a natural product $\Cup$, the shifted shuffle product.

Proposition 1. *Given* $\boldsymbol{\sigma}$ *and* $\boldsymbol{\tau}$ *two elements of* B'*, we have*

$$F_{\boldsymbol{\sigma}} \cdot F_{\boldsymbol{\tau}} = \sum_{\boldsymbol{\mu}\in\boldsymbol{\sigma}\Cup\boldsymbol{\tau}} F_{\boldsymbol{\mu}}. \tag{1.2}$$

We continue the analogy and show that the segmented composition algebra **SCQSym** defined in [9] is a subalgebra of **PSQSym**. In order to do this, we need the notion of the *recoils* of a partially signed permutation. We show that the ribbon basis $R_{\mathbf{I}}$ defined in [9] can be written as follows :

Proposition 2. *Let* $\mathbf{I}$ *be a segmented composition, we have*

$$R_{\mathbf{I}} = \sum_{\boldsymbol{\sigma}\in B';\ \mathrm{Rec}(\boldsymbol{\sigma})=I} F_{\boldsymbol{\sigma}}. \tag{1.3}$$

We then define a new basis $L_{\mathbf{I}}$ of **SCQSym** that is an analogue of the fundamental basis of Tevlin defined in [12]. This basis comes from a projection of **PSQSym** to **SCQSym** generalizing [5]. We define an equivalence relation on partially signed permutations where two signed permutations are equivalent if and only if their GC statistics are the same. We quotient **PSQSym** by this equivalence relation and define the basis $L_{\mathbf{I}}$ as the image of F_σ by this quotient. This basis, like the basis defined by Tevlin, has structure coefficients equal to binomial coefficients. This is Theorem 2 in Section 4.

Finally, we exhibit the change of basis between the bases $R_{\mathbf{I}}$ and $L_{\mathbf{J}}$ and show in Theorem 3 that the coefficients are positive integers and give a combinatorial interpretation in terms of partially signed permutations with fixed GC and recoils.

All the detailed definitions will be given in Section 2. In Section 3, we give the idea of the proof of Theorem 1 using Laguerre histories. In Section 4 we give the algebraic results in detail.

2 Notations and definitions

2.1 Signed permutations and segmented compositions

A *signed permutation* $\boldsymbol{\sigma}$ of size n is a permutation of n such that each value has a sign plus or minus. We denote by B_n the set of signed permutations of size n. We overline negative values and we say that $\bar{i} \in \boldsymbol{\sigma}$ if the value i has a negative sign in $\boldsymbol{\sigma}$. For example, $\boldsymbol{\sigma} = \bar{2}57836\bar{4}1$ is a signed permutation of size 8.

When we compare two values $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_j$ of a signed permutation $\boldsymbol{\sigma}$, we use the order $\bar{1} < 1 < \bar{2} < 2 < \cdots$.

In this paper we consider the set of signed permutations where 1 is not signed, which we call *partially signed permutations*. We denote by B'_n the set of these permutations. For example, $\boldsymbol{\sigma} = \bar{2}57836\bar{4}1$ is an element of B'_n.

A $31{-}2$ pattern of $\boldsymbol{\sigma}$ is a pair $(\sigma_i\sigma_{i+1}, \sigma_j)$ such that $j > i+1$ and $\boldsymbol{\sigma}_i > \boldsymbol{\sigma}_j > \boldsymbol{\sigma}_{i+1}$. We also consider $(31, \bar{2})$ patterns that are pairs $(\sigma_i\sigma_{i+1}, \bar{k})$ such that $\bar{k} \in \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_i \geq \bar{k} > \boldsymbol{\sigma}_{i+1}$. Note that in the second case the value $\bar{k}$ is not necessarily to the right of σ_i. For example, the $31{-}2$ patterns of $\boldsymbol{\sigma} = \bar{2}57836\bar{4}1$ are (83,6) and $(83,\bar{4})$ and the $(31,\bar{2})$ patterns of $\boldsymbol{\sigma}$ are $(83,\bar{4})$, $(\bar{4}1,\bar{2})$, and $(\bar{4}1,\bar{4})$).

A *segmented composition* [9] of an integer n is a finite sequence $\mathbf{I}$ of ℓ positive integers $(i_1, \ldots, i_\ell)$ that sum to n separated by vertical bars or commas. The *descent set* of $\mathbf{I}$ is the set of signed values $\mathrm{Des}(\mathbf{I}) = \{i_1,\ i_1+i_2, \ldots, i_1+\cdots+i_{\ell-1}\}$ where $i_1+i_2+\ldots+i_k$ is overlined if and only if there is a bar after i_k in $\mathbf{I}$. For example, $\mathrm{Des}(1|2|1,2,2) = \{\bar{1}, \bar{3}, 4, 6\}$. The *ADE-word* associated with $\mathbf{I}$ is the word w of size $n-1$ such that

$$\begin{aligned} \bar{i} \in \mathrm{Des}(\mathbf{I}) &\Rightarrow w_i = A; \\ i \in \mathrm{Des}(\mathbf{I}) &\Rightarrow w_i = E; \\ \bar{i} \notin \mathrm{Des}(\mathbf{I}), i \notin \mathrm{Des}(\mathbf{I}) &\Rightarrow w_i = D. \end{aligned} \tag{2.1}$$

We denote this word by $\mathrm{ade}(\mathbf{I})$. For example, $\mathrm{ade}(1|2|1,2,2) = ADAEDED$.

Let $\sigma \in B'_n$. Define the *recoil set* of σ as

$$\mathrm{RecSet}(\sigma) := \{i \mid \sigma_k \in \{i, \bar{i}\},\ \sigma_j = i+1 \Rightarrow j < k\} \cup \{\overline{i-1} \mid \bar{i} \in \sigma\}. \tag{2.2}$$

Note that if σ has no negative values then $\mathrm{RecSet}(\sigma)$ exactly corresponds to the set of values i such that $i+1$ is to its left, which is the usual definition of the recoil set on permutations.

The *recoil composition* $\mathrm{Rec}(\sigma)$ is the segmented composition whose descent set is $\mathrm{RecSet}(\sigma)$. For example, if $\sigma = \bar{2}57836\bar{4}1$ we have $\mathrm{RecSet}(\sigma) = \{\bar{1}, \bar{3}, 4, 6\}$ and $\mathrm{Rec}(\sigma) = (1|2|1,2,2)$.

Define the *Genocchi descent set* of a partially signed permutation σ of size n as

$$\mathrm{GDes}(\sigma) := \{i-1 \mid \sigma_j = i \Rightarrow \sigma_j > \sigma_{j+1}\} \cup \{\overline{i-1} \mid \bar{i} \in \sigma\}. \tag{2.3}$$

Define the *Genocchi composition of descents* $\mathrm{GC}(\sigma)$ as the segmented composition $\mathbf{I}$ whose descent set is $\mathrm{GDes}(\sigma)$. If σ is a permutation, the statistic GC is the set of values of descents minus one, as in [5]. For example, if $\sigma = \bar{2}57836\bar{4}1$, we have $\mathrm{GDes}(\sigma) = \{\bar{1}, \bar{3}, 5, 7\}$ and $\mathrm{GC}(\sigma) = (1|2|2,2,1)$.

Proposition 3. *We have* $\bar{i} \in \mathrm{GDes}(\sigma)$ *if and only if* $\bar{i} \in \mathrm{RecSet}(\sigma)$.

3 2-ASEP, Matrix Ansatz and Laguerre histories

The two-species ASEP is a Markov chain whose states are words of length N in the letters $\{0,1,2\}$. This was first studied in a more general setting in [13] and then in [2, 7]. Let x and y arbitrary words in $\{0,1,2\}^*$. The transitions in the Markov chain are the following:

$$x10y \to x01y;\ x20y \to x02y;\ x21y \to x12y;\ 0y \to 2y;\ y2 \to y0;$$

with rate 1 and

$$x01y \to x10y;\ x02y \to x20y;\ x12y \to x21y;$$

with rate q. Note that the number of 1's cannot change in the process and we denote it by r henceforth. An example of a chain on three letters among which two are equal to 1 is given on Figure 1.

3.1 Stationary distribution

To each state x of the 2-ASEP with N sites we associate a word $X(x)$ in $\{A, D, E\}^N$ using the following map:

$$2 \mapsto D;\ 1 \mapsto A;\ 0 \mapsto E.$$

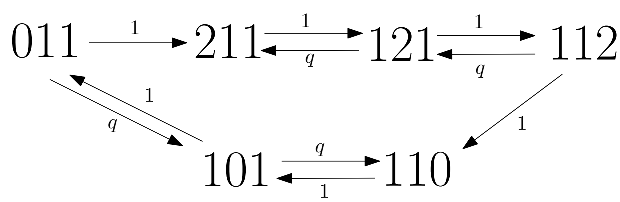

Figure 1: Chain for $N = 3$ and $r = 2$.

A segmented composition $\mathbf{I}$ codes the state x if $\mathrm{ade}(\mathbf{I}) = X(x)$.

The usual way to compute the stationary distribution of the states of the 2-ASEP is to find some matrices satisfying an Ansatz.

Proposition 4. *[13] Let D, A, E be infinite matrices. Let $\langle W|$ (respectively $|V\rangle$) be an infinite row (respectively column) vector satisfying the Ansatz:*

$$\begin{aligned} DE &= qED + D + E; \quad DA = qAD + A; \quad AE = qEA + A \\ \langle W| E &= \langle W|; \quad D|V\rangle = |V\rangle. \end{aligned}$$

Then the probability to be in a state x is:

$$\frac{\langle W| X(x) |V\rangle}{[y^r] \langle W| (D + yA + E)^N |V\rangle}. \tag{3.1}$$

where $[y^r]$ means that we consider the coefficient of the monomial y^r.

Remark 1. *The Ansatz in [13] is more general. Here, we set $\alpha = \beta = 1$ and $\gamma = \delta = 0$.*

We now give a new solution of this Ansatz. Let $[n]_q = 1 + q + \ldots + q^{n-1}$ and

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots \\ 0 & [2]_q & [2]_q & 0 & \ldots \\ 0 & 0 & [3]_q & [3]_q & \ldots \\ 0 & 0 & 0 & [4]_q & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}; \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 0 & [2]_q & [2]_q & 0 & \ldots \\ 0 & 0 & [3]_q & [3]_q & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.2}$$

$$A = \mathbf{diag}(1, q, q^2, q^3, \ldots)(D + E); \quad \langle W| = (1, 1, 0, 0, \ldots); \quad |V\rangle = (1, 0, 0, \ldots)^T. \tag{3.3}$$

Lemma 1. *The previous matrices and vectors satisfy the equations in Proposition 4.*

One way to obtain a combinatorial interpretation of the stationary distribution is to interpret each monomial of the numerator and denominator of (3.1) as a weighted path.

3.2 Path interpretation

A Laguerre history is an object introduced by Viennot in [14]. The marked Laguerre histories are in bijection with permutations through the Françon-Viennot bijection [4]. These objects have often been used to study some properties of the ASEP [6].

Definition 1. *A* Laguerre history *H of size n is a weighted Motzkin path of size n with two different horizontal steps such that*

- *a $\nearrow$ or $\longrightarrow$ step starting from height h has a weight between 0 and h;*
- *a $\searrow$ or $\dashrightarrow$ step starting from height h has a weight between 0 and $h-1$.*

The weight $\mathrm{wt}(H)$ *of the Laguerre history is the sum of the weights of its steps.*

A *marked Laguerre history* of size (n,r) is a Laguerre history of size n where all the steps but the first can be marked and r steps are marked. Any marked step starting from height h increases its weight by h.

An example of a marked Laguerre history of size $(8,2)$ is given in Figure 2. The steps with overlined weight are the marked steps. We can associate labels with a marked Laguerre history. The marked steps are labeled A, the $\nearrow$ or $\longrightarrow$ steps are labeled D and the remaining steps are labeled E. We forget the label of the first step. For example, the labels of the marked Laguerre history on the right of Figure 2 are represented by the word $ADADEDE$.

Given a word X, let $M(X)$ be the set of marked Laguerre histories with labels X and let $\mathcal{Z}_X$ be the generating polynomial of all the paths:

$$\mathcal{Z}_X(q) = \sum_{H \in M(X)} q^{\mathrm{wt}}(H); \tag{3.4}$$

and let

$$\mathcal{Z}_{N,r}(q) = \sum_X \mathcal{Z}_X(q) \tag{3.5}$$

where the sum is over all the words in $\{A, D, E\}^N$ with r letters A.

The following result gives us a combinatorial interpretation of (3.1).

Proposition 5. *Given a word X in $\{A, D, E\}^N$ with r letters A,*

$$\frac{\mathcal{Z}_X(q)}{\mathcal{Z}_{N,r}(q)}$$

is the stationary distribution of the state X in the 2-ASEP with N sites and r particles of type 1.

Proof. The idea is to associate a marked Laguerre history with each monomial of the matrix product of the numerator of (3.1). Any monomial corresponds to the product of N non zero coefficients $(X_k)_{i_k,j_k}$ where $X_k \in \{A, D, E\}$ is the matrix corresponding to the k-th letter of X. As the indices (i_k, j_k) must satisfy $i_k = j_{k-1}$, they can represent the successive heights of a path. Moreover, as our matrices in (3.2) are tridiagonal, $|i_k - j_k| \leq 1$ so the paths are Motzkin paths. In order to have a path starting from height 0, we need to add a $\nearrow$ or $\longrightarrow$ step at the beginning of the path depending on which coefficient of $\langle W|$ has been taken. For the steps labeled D, we have $j_k \in \{i_k, i_k + 1\}$ so the possible steps are $\nearrow$ or $\longrightarrow$, and for E, we have $j_k \in \{i_k, i_k - 1\}$ so the possible steps are $\searrow$ or $\dashrightarrow$.

The weight of the k-th step of the path corresponds to the power of q taken in the coefficient $(X_k)_{i_k,j_k}$. One can see that for the matrix D (steps $\nearrow$ or $\longrightarrow$) the possible weights are between 0 and i_k and that for the matrix E (steps $\searrow$ or $\dashrightarrow$) the possible weights are between 0 and $i_k - 1$. Finally for the matrix A, the weights are the same than for D and E on which we added i_k due to the $\mathbf{diag}(1, q, q^2, q^3, \ldots)$ factor. This proves that the paths we obtain are exactly the marked Laguerre histories. □

Special cases of Proposition 5 are:

$$\mathcal{Z}_{N,r}(1) = \binom{N}{r}(N+1)!; \quad \mathcal{Z}_{r,r}(q) = [r+1]_q!. \tag{3.6}$$

The first equation follows trivially from the next section which exhibits a bijection between these marked Laguerre histories and partially signed permutations. The second equation follows from a continued fraction proven by Heine. A bijective proof was given by Biane [1].

3.3 From marked Laguerre histories to partially signed permutations

We recall the Françon-Viennot bijection [4] that builds a Laguerre history from a permutation $\sigma \in \mathfrak{S}_n$. We denote this map by ψ_{FV}. We compare each value of the permutation σ with its two neighbors. We use the convention $\sigma_0 = 0$ and $\sigma_{n+1} = n + 1$. We denote by H_k the k-th step of the Laguerre history H.

Algorithm 1. *Let $\sigma \in \mathfrak{S}_n$, $k \in \{1, \ldots, n\}$, and j such that $\sigma_j = k$. Then, in the Laguerre history $H = \psi_{FV}(\sigma)$ we have*

- $H_k = \nearrow$ *if σ_j is a* valley, i.*e.*, $\sigma_{j-1} > \sigma_j < \sigma_{j+1}$,
- $H_k = \searrow$ *if σ_j is a* peak, i.*e.*, $\sigma_{j-1} < \sigma_j > \sigma_{j+1}$,
- $H_k = \longrightarrow$ *if σ_j is a* double rise, i.*e.*, $\sigma_{j-1} < \sigma_j < \sigma_{j+1}$,
- $H_k = \dashrightarrow$ *if σ_j is a* double descent, i.*e.*, $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$.

The weight w is constructed as follows: w_k is equal to the number of $31-2$ patterns such that k is the number corresponding to 2 in σ.

Now to generalize this to partially signed permutations, let $\sigma \in B'_n$. Apply the above bijection without considering the signs. For each $\bar{i} \in \sigma$, mark the i^{th} step of the Laguerre history. When we mark a step starting at height h, we increase its weight by h. We denote by $\psi_{FV}(\sigma)$ the result of this algorithm.

For example, for $\sigma = \bar{2}5783\bar{6}41$, the image of σ without the signs is given on the left of Figure 2 and the marked version is on its right.

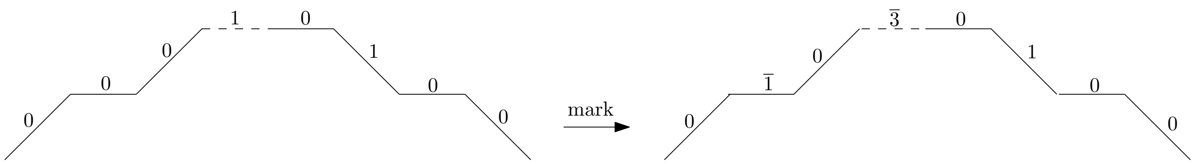

Figure 2: (Marked) Laguerre histories.

Proposition 6. *The generalization of the Françon-Viennot bijection is a bijection between partially signed permutations of size n with r overlined values and marked Laguerre histories of size n with r marked steps. The bijection is such that the number of $31-2$ patterns plus the number of $(31,\bar{2})$ patterns of the partially signed permutation is equal to the weight of the history. Moreover, if $\sigma \in B'_n$, then $\operatorname{ade}(\operatorname{GC}(\sigma))$ is the same as the labels of $\psi_{FV}(\sigma)$.*

To prove this proposition, one has to use the inverse of Françon-Viennot bijection and see that the patterns $(31,\bar{2})$ correspond to the $31-2$ patterns that would be created if $\bar{2}$ was also placed at the end of σ.

Using Proposition 5 and Proposition 6 we can deduce Theorem 1.

Remark 2. *If we take the inverse of the signed permutations in B' we obtain another subclass of B which are the signed permutations where the first value is not signed. The obtained statistic corresponds to the notion of ascent defined in [8]. The number of these permutations with r overlined values are $(r+1)!$ times the number of assemblées of permutations. Moreover, our polynomials $\mathcal{Z}_X(q)$ and $\mathcal{Z}_{n,r}(q)$ are equal to $[r+1]_q!$ times the polynomials of [8].*

4 Combinatorial algebras

The statistics GC on permutations is used to give a combinatorial interpretation of the stationary distribution of the ASEP [11]. This statistic also appeared in a totally different framework. When Tevlin defined a new basis of **Sym** [12], GC appeared in the computation of the transition matrices of this new basis to the ribbon basis. In this section we transpose the work of [5] into our context of partially signed permutations to generalize Tevlin's basis to the algebra of segmented compositions **SCQSym**.

4.1 The algebra of partially signed permutations

There are several ways to generalize the permutation algebra **FQSym**. One of them is the *signed permutation algebra* $\mathbf{FQSym}^{(2)}$ [10]. Here we define an algebra indexed by the partially signed permutations **PSQSym** as a subalgebra of $\mathbf{FQSym}^{(2)}$.

Since an element σ of B'_n does not have a sign on 1, this element can be viewed as the sum of two signed permutations $\overline{\sigma}$ and $\widetilde{\sigma}$ which correspond to σ where we respectively put a plus sign and a minus sign on 1.

Definition 2. *The algebra* **PSQSym** *is the subalgebra of* $\mathbf{FQSym}^{(2)}$ *spanned by the relations*

$$F_\sigma := \mathbf{F}_{\overline{\sigma}} + \mathbf{F}_{\widetilde{\sigma}}, \tag{4.1}$$

for $\sigma \in B'_n$ *where* $(\mathbf{F}_\tau)$ *is the fundamental basis of* $\mathbf{FQSym}^{(2)}$.

For a word w, denote by $w[k]$ the word obtained by replacing each letter i by the integer $i+k$. If σ and τ are two elements of B', one defines the *shifted shuffle* on partially signed permutations.

$$\sigma \Cup \tau = \sigma \sqcup\!\sqcup (\overline{\tau}[k]) + \sigma \sqcup\!\sqcup (\widetilde{\tau}[k]), \tag{4.2}$$

where k is the size of σ and $\sqcup\!\sqcup$ is the usual shuffle product on words defined by

$$(au) \sqcup\!\sqcup (bv) = a \cdot (u \sqcup\!\sqcup (bv)) + b \cdot ((au) \sqcup\!\sqcup v), \tag{4.3}$$

with $u \sqcup\!\sqcup \epsilon = \epsilon \sqcup\!\sqcup u = u$ if ϵ is the empty word.

The product formula of two F_σ given in Proposition 1 follows directly from the shifted shuffle over signed permutations.

4.2 The algebra of segmented compositions

In [9] the authors introduced the algebra of segmented compositions **SCQSym** generalizing the algebra of non commutative symmetric functions **Sym**. Indeed, when **Sym** can be viewed as the algebra of the vertices of the hypercube, **SCQSym** is the algebra of all the faces of the hypercube.

This algebra has a ribbon basis $(R_\mathbf{I})$ satisfying the following product formula

$$R_\mathbf{I} \cdot R_\mathbf{J} = R_{\mathbf{I}\cdot\mathbf{J}} + R_{\mathbf{I}|\mathbf{J}} + R_{\mathbf{I}\triangleright\mathbf{J}}. \tag{4.4}$$

Given $\mathbf{I} = (i_1, \ldots, i_s)$ and $\mathbf{J} = (j_1, \ldots, j_k)$ where the separators are either bars or commas, define

$$\begin{aligned} \mathbf{I}\cdot\mathbf{J} &:= (i_1, \ldots, i_s, j_1, \ldots, j_k), \\ \mathbf{I}|\mathbf{J} &:= (i_1, \ldots, i_s | j_1, \ldots, j_k), \\ \mathbf{I}\triangleright\mathbf{J} &:= (i_1, \ldots, i_s + j_1, \ldots, j_k). \end{aligned} \tag{4.5}$$

For example, $R_{(2|1)} \cdot R_{(1,3)} = R_{(2|1,1,3)} + R_{(2|1|1,3)} + R_{(2|2,3)}$.

Just as **Sym** is a subalgebra of **FQSym**, it can be shown that **SCQSym** is subalgebra of **PSQSym** algebra with the following identification given in Proposition 2.

$$R_{\mathbf{I}} = \sum_{\mathrm{Rec}(\sigma)=\mathbf{I}} F_\sigma \,. \tag{4.6}$$

4.3 Analogue of the basis of Tevlin

Let $\sim$ be the equivalence relation defined by $\sigma \sim \tau$ if and only if $\mathrm{GC}(\sigma) = \mathrm{GC}(\tau)$. Let $\mathcal{J}$ be the subspace of **PSQSym** spanned by the differences

$$\{F_\sigma - F_\tau \mid \sigma \sim \tau\}. \tag{4.7}$$

If **I** and **J** be two segmented compositions, we say that **I** is finer than **J** (or **J** is coarser than **I**) if and only if they have the same size, $\mathrm{Des}(\mathbf{I}) \subseteq \mathrm{Des}(\mathbf{J})$, and $\mathrm{Des}(\mathbf{I})$ has the same overlined values as $\mathrm{Des}(\mathbf{J})$.

Proposition 7. *$\mathcal{J}$ is a two-sided ideal of* **PSQSym**, *and the quotient* $\mathbf{T} = \mathbf{PSQSym}/\mathcal{J}$ *is isomorphic to* **SCQSym** *as an algebra.*

We have a map ϕ from **SCQSym** to itself that can be described as follows.

$$\begin{array}{ccccccc} \mathbf{SCQSym} & \hookrightarrow & \mathbf{PSQSym} & \twoheadrightarrow & \mathbf{T} & \leftrightarrow & \mathbf{SCQSym} \\ R_{\mathbf{I}} & \mapsto & \sum_{\mathrm{Rec}(\sigma)=\mathbf{I}} F_\sigma & & & & \\ & & F_\sigma & \mapsto & T_{\mathrm{GC}(\sigma)} & \mapsto & L_{\mathrm{GC}(\sigma)} \end{array} \tag{4.8}$$

This map ϕ is an isomorphism of algebras. Define $L_{\mathbf{I}}$ as the image of F_σ in **SCQSym**.

Theorem 2. *Let* **I** *and* **J** *be two segmented compositions,*

$$L_{\mathbf{I}} L_{\mathbf{J}} = \sum_{\mathbf{K}} C_{\mathbf{I},\mathbf{J}}^{\mathbf{K}} L_{\mathbf{K}}, \tag{4.9}$$

where $C_{\mathbf{I},\mathbf{J}}^{\mathbf{K}}$ is computed as follows. Let $\mathbf{K}'$ and $\mathbf{K}''$ be the segmented compositions such that $|\mathbf{K}'| = |\mathbf{I}|$ and either $\mathbf{K} = \mathbf{K}' \cdot \mathbf{K}''$, $\mathbf{K} = \mathbf{K}' \triangleright \mathbf{K}''$, or $\mathbf{K} = \mathbf{K}'|\mathbf{K}''$. If $\mathbf{K}'$ is not coarser than $\mathbf{I}$ or if $\mathbf{K}''$ is not finer than $\mathbf{J}$, then $C_{\mathbf{I},\mathbf{J}}^{\mathbf{K}}$ is 0. Otherwise, we have

$$C_{\mathbf{I},\mathbf{J}}^{\mathbf{K}} = \binom{|\mathbf{I}| + e(\mathbf{J}) - e(\mathbf{I}) + a(\mathbf{K}) - a(\mathbf{I})}{l(\mathbf{K}) - l(\mathbf{I})}, \tag{4.10}$$

where $a(\mathbf{I})$ is the number of bars in **I**, *$e(\mathbf{I})$ is the number of values in* **I** *minus the number of bars, and $l(\mathbf{I})$ is the number of values in* **I**.

The proof of this theorem follows the same ideas as the proof of Theorem 4.1. in [5].

4.4 Transition matrices

Identifying each $R_{\mathbf{I}}$ and its image under the map ϕ, we have the following result.

Theorem 3. *Let* $\mathbf{I}$ *be a segmented composition of* n*. Then*

$$R_{\mathbf{I}} = \sum_{\mathbf{J}} G_{\mathbf{IJ}} L_{\mathbf{J}}, \tag{4.11}$$

where $G_{\mathbf{IJ}}$ *is the number of partially signed permutations* $\boldsymbol{\sigma}$ *satisfying* $\mathrm{Rec}(\boldsymbol{\sigma}) = \mathbf{I}$ *and* $\mathrm{GC}(\boldsymbol{\sigma}) = \mathbf{J}$*. In particular, the* $G_{\mathbf{IJ}}$ *are non negative integers.*

In addition to this result, the transition matrices from R to L have some other interesting properties. Here is the transition matrix for $n = 3$. For better readability, 0 has been represented by a dot.

$$\left(\begin{array}{cccc|cc|cc|c}
1 & . & . & . & . & . & . & . & . \\
. & 2 & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
\hline
. & . & . & . & 3 & 1 & . & . & . \\
. & . & . & . & . & 2 & . & . & . \\
\hline
. & . & . & . & . & . & 2 & . & . \\
. & . & . & . & . & . & 1 & 3 & . \\
\hline
. & . & . & . & . & . & . & . & 6
\end{array}\right)$$

Proposition 3 implies that the matrix is block diagonal. It is also straightforward to see that the top left block is the transition matrix obtained in the permutation case [5] and that the sum of the values in each block is $n!$. It is also possible to prove that each value (except of the first block) are the sum of four other values in some blocks of bigger size.

5 Conclusion and acknowledgments

This work can be generalized in several ways. First we can define a monomial basis for **SCQSym** and define a $q-$analogue of some bases of this algebra. This uses the theory of segmented packed words. It gives an enumaration formula for $\mathcal{Z}_X(q)$.

The $2-$ASEP defined in [13] has four other parameters and two of them could also be interpreted in term of statistics on Laguerre histories. All this will be addressed in the long version of this extended abstract.

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